Chapter 11

Transients

11.1 STEADY STATE AND TRANSIENT RESPONSE

A circuit having constant sources is said to be in steady state if the currents and voltages do not change with time. Thus, circuits with currents and voltages having constant amplitude and constant frequency sinusoidal functions are also considered to be in a steady state. That means that the amplitude or frequency of a sinusoid never changes in a steady state circuit.

In a network containing energy storage elements, with change in excitation, the currents and voltages change from one state to other state. The behaviour of the voltage or current when it is changed from one state to another is called the *transient state*. The time taken for the circuit to change from one steady state to another steady state is called the *transient time*. The application of KVL and KCL to circuits containing energy storage elements results in differential, rather than algebraic, equations. When we consider a circuit containing storage elements which are independent of the sources, the response depends upon the nature of the circuit and is called the *natural response*. Storage elements deliver their energy to the resistances. Hence the response changes with time, gets saturated after some time, and is referred to as the *transient response*. When we consider sources acting on a circuit, the response depends on the nature of the source or sources. This response is called *forced response*. In other words, the complete response of a circuit consists of two parts: the forced response and the transient response. When we consider a differential equation, the complete solution consists of two parts: the complementary function and the particular solution. The complementary function dies out after short interval, and is referred to as the transient response or source free response. The particular solution is the steady state response, or the forced response. The first step in finding the complete solution of a circuit is to form a differential equation for the circuit. By obtaining the differential equation, several methods can be used to find out the complete solution.

11.2 DC RESPONSE OF AN R-L CIRCUIT

Consider a circuit consisting of a resistance and inductance as shown in Fig. 11.1. The inductor in the circuit is initially uncharged and is in series with the resistor. When the switch S is closed, we can find the complete solution for the current. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

Fig. 11.1

$$V = Ri + L\frac{di}{dt} \tag{11.1}$$

or

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L} \tag{11.2}$$

In the above equation, the current i is the solution to be found and V is the applied constant voltage. The voltage V is applied to the circuit only when the switch S is closed. The above equation is a linear differential equation of first order. Comparing it with a non-homogeneous differential equation

$$\frac{dx}{dt} + Px = K \tag{11.3}$$

whose solution is

$$x = e^{-pt}\int Ke^{+Pt}\,dt + ce^{-Pt} \tag{11.4}$$

where c is an arbitrary constant. In a similar way, we can write the current equation as

$$i = ce^{-(R/L)t} + e^{-(R/L)t}\int \frac{V}{L}e^{(R/L)t}\,dt$$

$$\therefore \quad i = ce^{-(R/L)t} + \frac{V}{R} \tag{11.5}$$

To determine the value of c in Eq. 11.5, we use the initial conditions. In the circuit shown in Fig. 11.1, the switch S is closed at $t = 0$. At $t = 0^-$, i.e. just before closing the switch S, the current in the inductor is zero. Since the inductor does not allow sudden changes in currents, at $t = 0^+$ just after the switch is closed, the current remains zero.

Thus at

$$t = 0, i = 0$$

Substituting the above condition in Eq. 11.5, we have

$$0 = c + \frac{V}{R}$$

Hence $$c = -\frac{V}{R}$$

Substituting the value of c in Eq. 5, we get

$$i = \frac{V}{R} - \frac{V}{R}\exp\left(-\frac{R}{L}t\right)$$

$$i = \frac{V}{R}\left(1 - \exp\left(-\frac{R}{L}t\right)\right) \qquad (11.6)$$

Equation 11.6 consists of two parts, the steady state part V/R, and the transient part $(V/R)e^{-(R/L)t}$. When switch S is closed, the response reaches a steady state value after a time interval as shown in Fig. 11.2.

Here the transition period is defined as the time taken for the current to reach its final or steady state value from its initial value. In the transient part of the solution, the quantity L/R is important in describing the curve since L/R is the time required for the current to reach from its initial value of zero to the final value V/R. The time constant of a function $\frac{V}{R}e^{-\left(\frac{R}{L}\right)t}$ is the time at which the exponent of e is unity, where e is the base of the natural logarithms. The term L/R is called the *time constant* and is denoted by τ

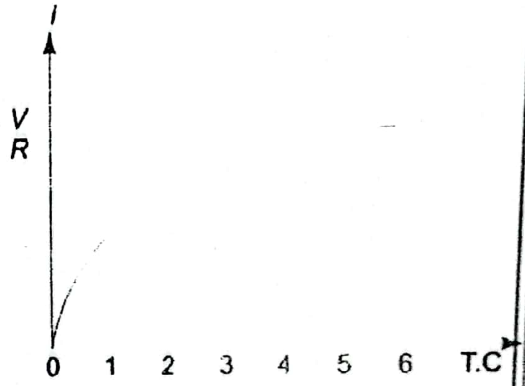

Fig. 11.2

$$\therefore \qquad \tau = \frac{L}{R} \text{ sec}$$

$\therefore$ The transient part of the solution is

$$i = -\frac{V}{R}\exp\left(-\frac{R}{L}t\right) = -\frac{V}{R}e^{-t/\tau}$$

At one TC, i.e. at one time constant, the transient term reaches 36.8 percent of its initial value.

$$i(\tau) = -\frac{V}{R}e^{-t/\tau} = -\frac{V}{R}e^{-1} = -0.368\frac{V}{R}$$

Similarly,

$$i(2\tau) = -\frac{V}{R}e^{-2} = -0.135\frac{V}{R}$$

$$i(3\tau) = -\frac{V}{R}e^{-3} = -0.0498\ \frac{V}{R}$$

$$i(5\tau) = -\frac{V}{R}e^{-5} = -0.0067\ \frac{V}{R}$$

After 5 TC, the transient part reaches more than 99 percent of its final va

In Fig. 11.1, we can find out the voltages and powers across each elemen using the current.

Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R}\left[1 - \exp\left(-\frac{R}{L}t\right)\right]$$

$$\therefore \quad v_R = V\left[1 - \exp\left(-\frac{R}{L}t\right)\right]$$

Similarly, the voltage across the inductance is

$$v_L = L\frac{di}{dt}$$

$$= L\frac{V}{R} \times \frac{R}{L}\exp\left(-\frac{R}{L}t\right) = V\exp\left(-\frac{R}{L}t\right)$$

The response are shown in Fig. 11.3

Power in the resistor is

$$p_R = v_R i = V\left(1 - \exp\left(-\frac{R}{L}t\right)\right)\left(1 - \exp\left(-\frac{R}{L}t\right)\right)\frac{V}{R}$$

$$= \frac{V^2}{R}\left(1 - 2\exp\left(-\frac{R}{L}t\right) + \exp\left(-\frac{2R}{L}t\right)\right)$$

Power in the inductor is

$$p_L = v_L i = V\exp\left(-\frac{R}{L}t\right) \times \frac{V}{R}\left(1 - \exp\left(-\frac{R}{L}t\right)\right)$$

$$= \frac{V^2}{R}\left(\exp\left(-\frac{R}{L}t\right) - \exp\left(-\frac{2R}{L}t\right)\right)$$

The responses are shown in Fig. 11.4.

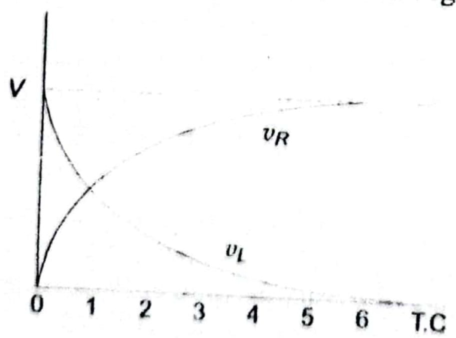

Fig. 11.3

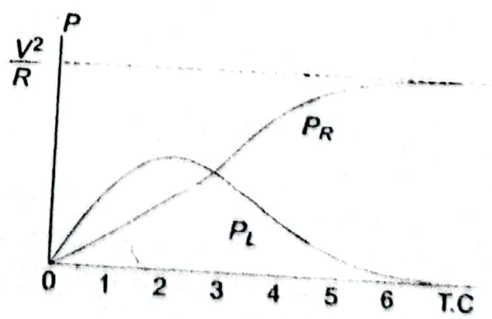

Fig. 11.4

Example 11.1 A series RL circuit with $R = 30\ \Omega$ and $L = 15$ H has a constant voltage $V = 60$ V applied at $t = 0$ as shown in Fig. 11.5. Determine the current i, the voltage across resistor and the voltage across the inductor.

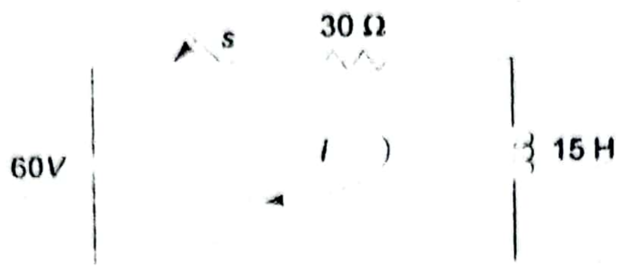

Fig. 11.5

Solution By applying Kirchhoff's voltage law, we get

$$15\frac{di}{dt} + 30i = 60$$

$$\frac{di}{dt} + 2i = 4$$

The general solution for a linear differential equation is

$$i = ce^{-Pt} + e^{-Pt}\int Ke^{Pt}\,dt$$

where $P = 2$, $K = 4$

$$i = ce^{-2t} + e^{-2t}\int 4e^{2t}\,dt$$

$$i = ce^{-2t} + 2$$

At $t = 0$, the switch S is closed.

∴ Since the inductor never allows sudden changes in currents. At $t = 0^+$ the current in the circuit is zero.

Therefore at $t = 0^+$, $i = 0$

∴ $0 = c + 2$

∴ $c = -2$

Substituting the value of c in the current equation, we have

$$i = 2(1 - e^{-2t})\text{ A}$$

Voltage across resistor $v_R = iR$

$$= 2(1 - e^{-2t}) \times 30 = 60\,(1 - e^{-2t})\text{ V}$$

Voltage across inductor $v_L = L\dfrac{di}{dt}$

$$= 15 \times \frac{d}{dt}\,2(1 - e^{-2t}) = 30 \times 2e^{-2t} = 60e^{-2t}\text{ V}$$

11.3 DC RESPONSE OF AN R-C CIRCUIT

Consider a circuit consisting of resistance and capacitance as shown in Fig. 11.6. The capacitor in the circuit is initially uncharged, and is in series with a resistor. When the switch S is closed at $t = 0$, we can determine the complete solution for the current. Application of the Kirchhoff's voltage law to the circuit results in the following differential equation.

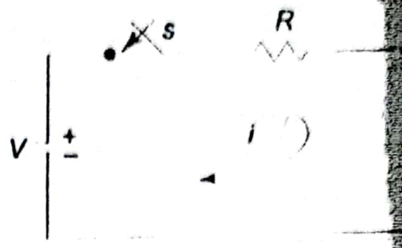

Fig. 11.6

$$V = Ri + \frac{1}{C}\int i\, dt \quad (11.7)$$

By differentiating the above equation, we get

$$0 = R\frac{di}{dt} + \frac{i}{C} \quad (11.8)$$

or
$$\frac{di}{dt} + \frac{1}{RC}i = 0 \quad (11.9)$$

Equation 11.9 is a linear differential equation with only the complementary function. The particular solution for the above equation is zero. The solution for this type of differential equation is

$$i = ce^{-t/RC} \quad (11.10)$$

Here, to find the value of c, we use the initial conditions.

In the circuit shown in Fig. 11.6, switch S is closed at $t = 0$. Since the capacitor never allows sudden changes in voltage, it will act as a short circuit at $t = 0^+$. So, the current in the circuit at $t = 0^+$ is V/R

$\therefore$ At $t = 0$, the current $i = \dfrac{V}{R}$

Substituting this current in Eq. 11.10, we get

$$\frac{V}{R} = c$$

$\therefore$ The current equation becomes

$$i = \frac{V}{R}e^{-t/RC} \quad (11.11)$$

When switch S is closed, the response decays with time as shown in Fig. 11.7.

In the solution, the quantity RC is the time constant, and is denoted by τ,

where $\tau = RC$ sec

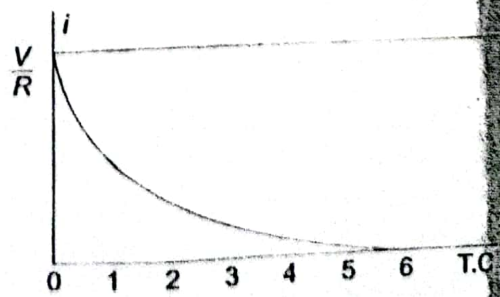

Fig. 11.7

After 5 TC, the curve reaches 99 per cent of its final value. In Fig. 11.6, we find out the voltage across each element by using the current equation.

Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R} e^{-(1/RC)t} ; v_R = Ve^{-\frac{t}{RC}}$$

Similarly, voltage across the capacitor is

$$v_C = \frac{1}{C} \int i\,dt$$

$$= \frac{1}{C} \int \frac{V}{R} e^{-t/RC} dt$$

$$= -\left(\frac{V}{RC} \times RC\,e^{-t/RC} \right) + c = -Ve^{-t/RC} + c$$

At $t = 0$, voltage across capacitor is zero

$$c = V$$

$$v_C = V(1 - e^{-t/RC})$$

The responses are shown in Fig. 11.8.

Power in the resistor

$$p_R = v_R i = Ve^{-t/RC} \times \frac{V}{R} e^{-t/RC} = \frac{V^2}{R} e^{-2t/RC}$$

Power in the capacitor

$$p_C = v_C i = V(1 - e^{-t/RC}) \frac{V}{R} e^{-t/RC}$$

$$= \frac{V^2}{R} (e^{-t/RC} - e^{-2t/RC})$$

The responses are shown in Fig. 11.9.

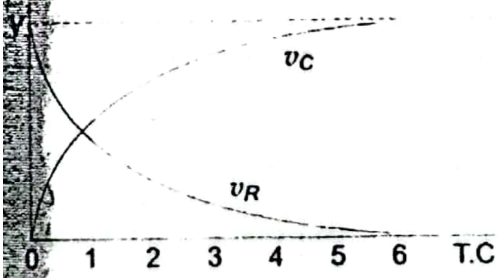

Fig. 11.8

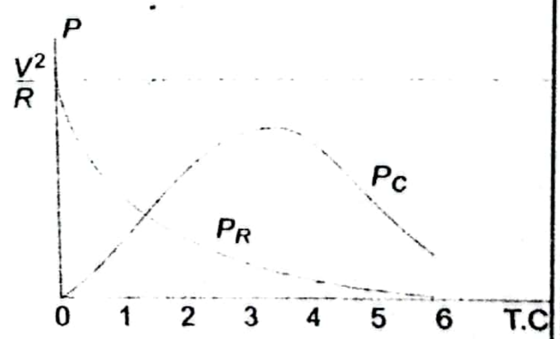

Fig. 11.9

Example 11.2 A series RC circuit consists of resistor of 10 Ω and capacitor of 0.1 F as shown in Fig. 11.10. A constant voltage of 20 V is applied to the circuit at $t = 0$. Obtain the current equation. Determine the voltages across the resistor and the capacitor.

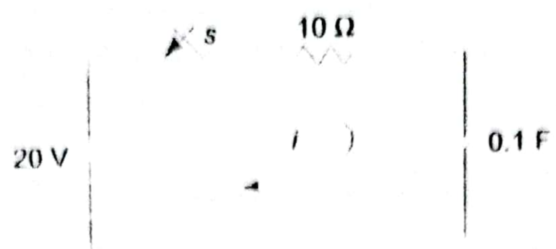

Fig. 11.10

Solution BY applying Kirchhoff's law, we get

$$10i + \frac{1}{0.1}\int i\,dt = 20$$

Differentiating with respect to t we get

$$10\frac{di}{dt} + \frac{i}{0.1} = 0$$

$$\therefore \quad \frac{di}{dt} + i = 0$$

The solution for the above equation is $i = ce^{-t}$

At $t = 0$, switch S is closed. Since the capacitor does not allow sudden changes in voltage, the current in the circuit is $i = V/R = 20/10 = 2$ A.

At $t = 0$, $i = 2$ A.

$\therefore$ The current equation $i = 2e^{-t}$

Voltage across the resistor is $v_R = i \times R = 2e^{-t} \times 10 = 20e^{-t}$ V

Voltage across the capacitor is $v_C = V\left(1 - e^{-\frac{t}{RC}}\right)$

$$= 20\,(1 - e^{-t})\text{ V}$$

11.4 DC RESPONSE OF AN R-L-C CIRCUIT

Consider a circuit consisting of resistance, inductance and capacitance as shown in Fig. 11.11. The capacitor and inductor are initially uncharged, and are in series with a resistor. When switch S is closed at $t = 0$, we can determine the complete solution for the current. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

Fig. 11.11

$$V = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \tag{11.12}$$

By differentiating the above equation, we have

$$0 = R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C}i \tag{11.13}$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0 \qquad (11.14)$$

The above equation is a second order linear differential equation, with only complementary function. The particular solution for the above equation is zero. Characteristic equation for the above differential equation is

$$\left(D^2 + \frac{R}{L} D + \frac{1}{LC}\right) = 0 \qquad (11.15)$$

The roots of Eq. 11.15 are

$$D_1, D_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

By assuming $K_1 = -\frac{R}{2L}$ and $K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

$$D_1 = K_1 + K_2 \text{ and } D_2 = K_1 - K_2$$

Here K_2 may be positive, negative or zero.

K_2 is positive, when $\left(\frac{R}{2L}\right)^2 > 1/LC$

The roots are real and unequal, and give the over damped response as shown in Fig. 11.12. Then Eq. 11.14 becomes

$$[D - (K_1 + K_2)]\,[D - (K_1 - K_2)]\, i = 0$$

The solution for the above equation is

$$i = c_1 e^{(K_1 + K_2)t} + c_2 e^{(K_1 - K_2)t}$$

The current curve for the overdamped case is shown in Fig. 11.12.

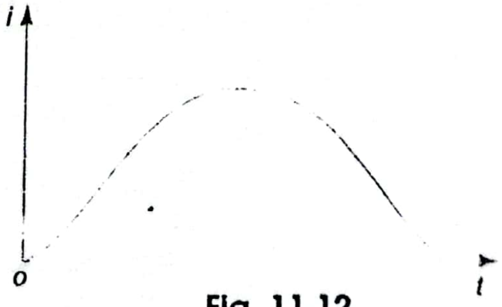

Fig. 11.12

K_2 is negative, when $(R/2L)^2 < 1/LC$

The roots are complex conjugate, and give the underdamped response as shown in Fig. 11.13. Then Eq. 11.14 becomes

$$[D - (K_1 + jK_2)]\,[D - (K_1 - jK_2)]i = 0$$

The solution for the above equation is

$$i = e^{K_1 t}\,[c_1 \cos K_2 t + c_2 \sin K_2 t]$$

The current curve for the underdamped case is shown in Fig. 11.13.

K_2 is zero, when $(R/2L)^2 = 1/LC$

The roots are equal, and give the critically damped response as shown in Fig. 11.14. Then Eq. 11.14 becomes

$$(D - K_1)\,(D - K_1)i = 0$$

The solution for the above equation is

$$i = e^{K_1 t}\,(c_1 + c_2 t)$$

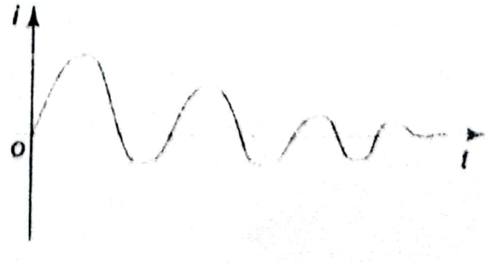

Fig. 11.13

The current curve for the critically damped case is shown in Fig. 11.14.

Fig. 11.14

Example 11.3 The circuit shown in Fig. 11.15 consists of resistance, inductance and capacitance in series with a 100 V constant source when the switch is closed at $t = 0$. Find the current transient.

Solution At $t = 0$, switch S is closed when the 100 V source is applied to the circuit and results in the following differential equation.

$$100 = 20i + 0.05\frac{di}{dt} + \frac{1}{20 \times 10^{-6}}\int i\,dt \quad (11.16)$$

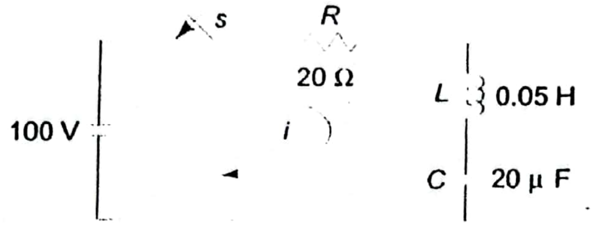

Fig. 11.15

Differentiating the Eq. 11.16, we get

$$0.05\frac{d^2i}{dt^2} + 20\frac{di}{dt} + \frac{1}{20 \times 10^{-6}}i = 0$$

$$\frac{d^2i}{dt^2} + 400\frac{di}{dt} + 10^6 i = 0$$

$$\therefore \quad (D^2 + 400D + 10^6)i = 0$$

$$D_1, D_2 = -\frac{400}{2} \pm \sqrt{\left(\frac{400}{2}\right)^2 - 10^6}$$

$$= -200 \pm \sqrt{(200)^2 - 10^6}$$

$$D_1 = -200 + j979.8$$

$$D_2 = -200 - j979.8$$

Therefore the current

$$i = e^{+k_1 t}[c_1 \cos K_2 t + c_2 \sin K_2 t)]$$

$$i = e^{-200t}[c_1 \cos 979.8t + c_2 \sin 979.8t)] \text{ A}$$

At $t = 0$, the current flowing through the circuit is zero

$$i = 0 = (1)[c_1 \cos 0 + c_2 \sin 0]$$

$$c_1 = 0$$

$$i = e^{-200t} c_2 \sin 979.8t \text{ A}$$

Differentiating, we have

$$\frac{di}{dt} = c_2 [e^{-200t}\, 979.8 \cos 979.8t + e^{-200t} (-200) \sin 979.8t)]$$

At $t = 0$, the voltage across inductor is 100 V

$$L \frac{di}{dt} = 100$$

or $$\frac{di}{dt} = 2000$$

At $t = 0$ $$\frac{di}{dt} = 2000 = c_2\, 979.8 \cos 0$$

$$\therefore \quad c_2 = \frac{2000}{979.8} = 2.04$$

The current equation is

$$i = e^{-200t} (2.04 \sin 979.8t) \text{ A}$$

11.5 SINUSOIDAL RESPONSE OF R-L CIRCUIT

Consider a circuit consisting of resistance and inductance as shown in Fig. 11.16. The switch, S, is closed at $t = 0$. At $t = 0$, a sinusoidal voltage $V \cos (\omega t + \theta)$ is applied to the series R-L circuit, where V is the amplitude of the wave and θ is the phase angle. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

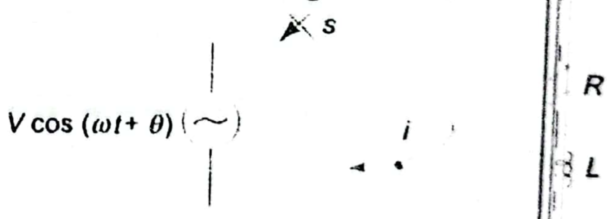

Fig. 11.16

$$V \cos (\omega t + \theta) = R i + L \frac{di}{dt} \tag{11.17}$$

$$\therefore \quad \frac{di}{dt} + \frac{R}{L} i = \frac{V}{L} \cos (\omega t + \theta)$$

The corresponding characteristic equation is

$$\left(D + \frac{R}{L}\right) i = \frac{V}{L} \cos (\omega t + \theta) \tag{11.18}$$

For the above equation, the solution consists of two parts, viz. complementary function and particular integral.

The complementary function of the solution i is

$$i_c = c e^{-t(R/L)} \tag{11.19}$$

The particular solution can be obtained by using undetermined co-efficients.

By assuming $$i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta) \quad (11.20)$$
$$i'_p = -A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta) \quad (11.21)$$

Substituting Eqs 11.20 and 11.21 in Eq. 11.18, we have

$$\{-A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta) + \frac{R}{L}\{A\cos(\omega t + \theta)$$
$$+ B\sin(\omega t + \theta)\}\} = \frac{V}{L}\cos(\omega t + \theta)$$

or $$\left(-A\omega + \frac{BR}{L}\right)\sin(\omega t + \theta) + \left(B\omega + \frac{AR}{L}\right)\cos(\omega t + \theta) = \frac{V}{L}\cos(\omega t + \theta)$$

Comparing cosine terms and sine terms, we get

$$-A\omega + \frac{BR}{L} = 0$$
$$B\omega + \frac{AR}{L} = \frac{V}{L}$$

From the above equations, we have

$$A = V\frac{R}{R^2 + (\omega L)^2}$$
$$B = V\frac{\omega L}{R^2 + (\omega L)^2}$$

Substituting the values of A and B in Eq. 11.20, we get

$$i_p = V\frac{R}{R^2 + (\omega L)^2}\cos(\omega t + \theta) + V\frac{\omega L}{R^2 + (\omega L)^2}\sin(\omega t + \theta) \quad (11.22)$$

Putting $$M\cos\phi = \frac{VR}{R^2 + (\omega L)^2}$$

and $$M\sin\phi = V\frac{\omega L}{R^2 + (\omega L)^2}.$$

to find M and ϕ, we divide one equation by the other

$$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{\omega L}{R}$$

Squaring both equations and adding, we get

$$M^2\cos^2\phi + M^2\sin^2\phi = \frac{V^2}{R^2 + (\omega L)^2}$$

or $$M = \frac{V}{\sqrt{R^2 + (\omega L)^2}}$$

The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}}\cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right) \quad (11.23)$$

The complete solution for the current $i = i_c + i_p$

$$i = ce^{-t(R/L)} + \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

Since the inductor does not allow sudden changes in currents, at $t = 0$, $i = 0$

$$\therefore \quad c = -\frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1}\frac{\omega L}{R}\right)$$

The complete solution for the current is

$$i = e^{-(R/L)t}\left[\frac{-V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1}\frac{\omega L}{R}\right)\right]$$

$$+ \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

Example 11.4 In the circuit shown in Fig. 11.17, determine the complete solution for the current, when switch S is closed at $t = 0$. Applied voltage is $v(t) = 100 \cos(10^3 t + \pi/2)$. Resistance $R = 20\ \Omega$ and inductance $L = 0.1$ H.

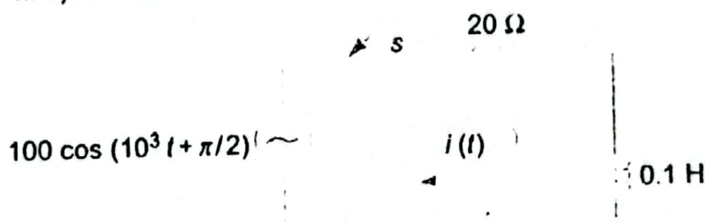

Fig. 11.17

Solution By applying Kirchhoff's voltage law to the circuit, we have

$$20i + 0.1\frac{di}{dt} = 100 \cos(10^3 t + \pi/2)$$

$$\frac{di}{dt} + 200i = 1000 \cos(1000t + \pi/2)$$

$$(D + 200)i = 1000 \cos(1000t + \pi/2)$$

The complementary function $i_c = ce^{-200t}$

By assuming particular integral as

$$i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta)$$

we get

$$i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

where $\omega = 1000$ rad/sec $V = 100$ V

$$\theta = \pi/2$$

$$L = 0.1\ \text{H},\ R = 20\ \Omega$$

Substituting the values in the above equation, we get

$$i_p = \frac{100}{\sqrt{(20)^2 + (1000 \times 0.1)^2}} \cos\left(1000t + \frac{\pi}{2} - \tan^{-1}\frac{100}{20}\right)$$

$$= \frac{100}{101.9} \cos\left(1000t + \frac{\pi}{2} - 78.6^\circ\right)$$

$$= 0.98 \cos\left(1000t + \frac{\pi}{2} - 78.6^\circ\right)$$

The complete solution is

$$i = ce^{-200t} + 0.98 \cos\left(1000t + \frac{\pi}{2} - 78.6^\circ\right)$$

At $t = 0$, the current flowing through the circuit is zero, i.e. $i = 0$

$$c = -0.98 \cos\left(\frac{\pi}{2} - 78.6^\circ\right)$$

The complete solution is

$$i = \left[-0.98 \cos\left(\frac{\pi}{2} - 78.6^\circ\right)\right] e^{-200t} + 0.98 \cos\left(1000t + \frac{\pi}{2} - 78.6^\circ\right)$$

11.6 SINUSOIDAL RESPONSE OF R-C CIRCUIT

Consider a circuit consisting of resistance and capacitance in series as shown in Fig. 11.18. The switch, S, is closed at $t = 0$. At $t = 0$, a sinusoidal voltage $V \cos(\omega t + \theta)$ is applied to the R-C circuit, where V is the amplitude of the wave and θ is the phase angle. Applying Kirchhoff's voltage law to the circuit results in the following differential equation.

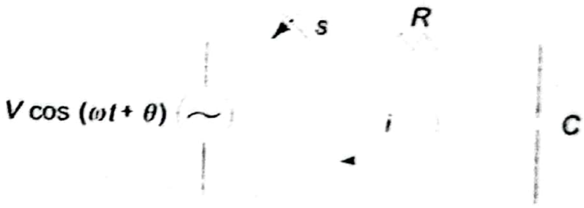

Fig. 11.18

$$V \cos(\omega t + \theta) = Ri + \frac{1}{C}\int i\,dt \qquad (11.24)$$

$$R\frac{di}{dt} + \frac{i}{C} = -V\omega \sin(\omega t + \theta)$$

$$\left(D + \frac{1}{RC}\right)i = -\frac{V\omega}{R}\sin(\omega t + \theta) \qquad (11.25)$$

The complementary function $i_C = ce^{-t/RC}$ (11.26)

The particular solution can be obtained by using undetermined coefficients.

$$i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta) \qquad (11.27)$$

$$i_p' = -A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta) \qquad (11.28)$$

Substituting Eqs 11.27 and 11.28 in Eq. 11.25, we get

$$\{-A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta)\}$$

$$+ \frac{1}{RC}\{A\cos(\omega t + \theta) + B\sin(\omega t + \theta)\}$$

$$= -\frac{V\omega}{R}\sin(\omega t+\theta)$$

Comparing both sides, $$-A\omega+\frac{B}{RC}=-\frac{V\omega}{R}$$

$$B\omega+\frac{A}{RC}=0$$

From which,

$$A=\frac{VR}{R^2+\left(\frac{1}{\omega c}\right)^2}$$

and $$B=\frac{-V}{\omega C\left[R^2+\left(\frac{1}{\omega c}\right)^2\right]}$$

Substituting the values of A and B in Eq. 11.27, we have

$$i_p=\frac{VR}{R^2+\left(\frac{1}{\omega c}\right)^2}\cos(\omega t+\theta)+\frac{-V}{\omega C\left[R^2+\left(\frac{1}{\omega C}\right)^2\right]}\sin(\omega t+$$

Putting $$M\cos\phi=\frac{VR}{\left[R^2+\left(\frac{1}{\omega C}\right)^2\right]}$$

and $$M\sin\phi=\frac{V}{\omega C\left[R^2+\left(\frac{1}{\omega C}\right)^2\right]}$$

To find M and ϕ, we divide one equation by the other,

$$\frac{M\sin\phi}{M\cos\phi}=\tan\phi=\frac{1}{\omega CR}$$

Squaring both equations and adding, we get

$$M^2\cos^2\phi+M^2\sin^2\phi=\frac{V^2}{\left[R^2+\left(\frac{1}{\omega C}\right)^2\right]}$$

$$\therefore \quad M=\frac{V}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}}$$

The particular current becomes

$$i_p=\frac{V}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}}\cos\left(\omega t+\theta+\tan^{-1}\frac{1}{\omega CR}\right)$$

The complete solution for the current $i = i_c + i_p$

$$i = ce^{-(t/RC)} + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right) \quad (11.30)$$

Since the capacitor does not allow sudden changes in voltages at $t = 0$, $i = \frac{V}{R}$ $\cos\theta$

$$\frac{V}{R}\cos\theta = c + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$c = \frac{V}{R}\cos\theta - \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

The complete solution for the current is

$$i = e^{-(t/RC)}\left[\frac{V}{R}\cos\theta - \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)\right]$$

$$+ \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right) \quad (11.31)$$

Example 11.5 In the circuit shown in Fig. 11.19, determine the complete solution for the current when switch S is closed at $t = 0$. Applied voltage is $v(t) = 50\cos\left(10^2 t + \frac{\pi}{4}\right)$. Resistance $R = 10\ \Omega$ and capacitance $C = 1\ \mu F$.

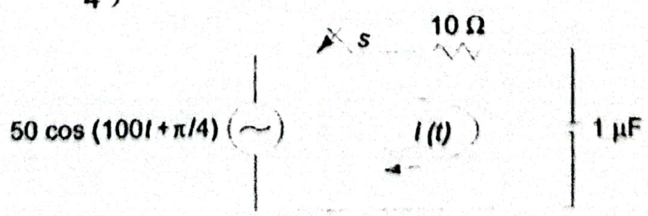

Fig. 11.19

Solution By applying Kirchhoff's voltage law to the circuit, we have

$$10i + \frac{1}{1\times 10^{-6}}\int i\,dt = 50\cos\left(100t + \frac{\pi}{4}\right)$$

$$10\frac{di}{dt} + \frac{i}{1 \times 10^{-6}} = -5(10)^3 \sin\left(100t + \frac{\pi}{4}\right)$$

$$\frac{di}{dt} + \frac{i}{10^{-5}} = -500 \sin\left(100t + \frac{\pi}{4}\right)$$

$$\left(D + \frac{1}{10^{-5}}\right) i = -500 \sin\left(100t + \frac{\pi}{4}\right)$$

The complementary function is $i_C = ce^{-t/10^{-5}}$. By assuming particular integral as $i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta)$,

we get
$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

where $\omega = 100$ rad/sec $\qquad \theta = \pi/4$

$C = 1\mu F \qquad R = 10\ \Omega$

Substituting the values in the above equation, we have

$$i_p = \frac{50}{\sqrt{(10)^2 + \left(\frac{1}{100 \times 10^{-6}}\right)^2}} \cos\left(\omega t + \frac{\pi}{4} + \tan^{-1}\frac{1}{100 \times 10^{-6} \times 10}\right)$$

$$i_p = 4.99 \times 10^{-3} \cos\left(100t + \frac{\pi}{4} + 89.94^\circ\right)$$

At $t = 0$, the current flowing through the circuit is

$$\frac{V}{R}\cos\theta = \frac{50}{10}\cos\pi/4 = 3.53 \text{ A}$$

$$i = \frac{V}{R}\cos\theta = 3.53 \text{ A}$$

$\therefore$
$$i = ce^{-t/10^{-5}} + 4.99 \times 10^{-3} \cos\left(100t + \frac{\pi}{4} + 89.94^\circ\right)$$

At $t = 0$

$$c = 3.53 - 4.99 \times 10^{-3} \cos\left(\frac{\pi}{4} + 89.94^\circ\right)$$

Hence the complete solution is

$$i = \left[3.53 - 4.99 \times 10^{-3} \cos\left(\frac{\pi}{4} + 89.94^\circ\right)\right] e^{-(t/10^{-5})}$$

$$+ 4.99 \times 10^{-3} \cos\left(100t + \frac{\pi}{4} + 89.94^\circ\right)$$

11.7 SINUSOIDAL RESPONSE OF R-L-C CIRCUIT

Consider a circuit consisting of resistance, inductance and capacitance in series as shown in Fig. 11.20. Switch S is closed at $t = 0$. At $t = 0$, a sinusoidal voltage

$V\cos(\omega t+\theta)$ is applied to the RLC series circuit, where V is the amplitude of the wave and θ is the phase angle. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

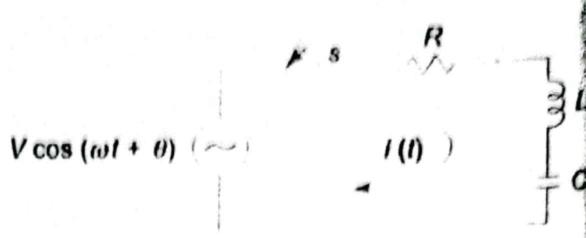

Fig. 11.20

$$V\cos(\omega t+\theta)=Ri+L\frac{di}{dt}+\frac{1}{C}\int i\,dt \tag{11.32}$$

Differentiating the above equation, we get

$$R\frac{di}{dt}+L\frac{d^2i}{dt^2}+i/C=-V\omega\sin(\omega t+\theta)$$

$$\left(D^2+\frac{R}{L}D+\frac{1}{LC}\right)i=-\frac{V\omega}{L}\sin(\omega t+\theta) \tag{11.33}$$

The particular solution can be obtained by using undetermined coefficients. By assuming

$$i_p=A\cos(\omega t+\theta)+B\sin(\omega t+\theta) \tag{11.34}$$

$$i'_p=-A\omega\sin(\omega t+\theta)+B\omega\cos(\omega t+\theta) \tag{11.35}$$

$$i''_p=-A\omega^2\cos(\omega t+\theta)-B\omega^2\sin(\omega t+\theta) \tag{11.36}$$

Substituting ip, i'_p and i''_p in Eq. 11.33, we have

$$\{-A\omega^2\cos(\omega t+\theta)-B\omega^2\sin(\omega t+\theta)\}$$

$$+\frac{R}{L}\{-A\omega\sin(\omega t+\theta)+B\omega\cos(\omega t+\theta)\}$$

$$+\frac{1}{LC}\{A\cos(\omega t+\theta)+B\sin(\omega t+\theta)\}=-\frac{V\omega}{L}\sin(\omega t+\theta) \tag{11.37}$$

Comparing both sides, we have

Sine coefficients.

$$-B\omega^2-A\frac{\omega R}{L}+\frac{B}{LC}=-\frac{V\omega}{L}$$

$$A\left(\frac{\omega R}{L}\right)+B\left(\omega^2-\frac{1}{LC}\right)=\frac{V\omega}{L} \tag{11.38}$$

Cosine coefficients

$$-A\omega^2+B\frac{\omega R}{L}+\frac{A}{LC}=0$$

$$A\left(\omega^2-\frac{1}{LC}\right)-B\left(\frac{\omega R}{L}\right)=0 \tag{11.39}$$

Solving Eqs 11.38 and 11.39, we get

$$A = \frac{V \times \dfrac{\omega^2 R}{L^2}}{\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]}$$

$$B = \frac{\left(\omega^2 - \dfrac{1}{LC}\right) V\omega}{L\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]}$$

Substituting the values of A and B in Eq. 11.34, we get

$$i_p = \frac{V \dfrac{\omega^2 R}{L^2}}{\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]} \cos(\omega t + \theta)$$

$$+ \frac{\left(\omega^2 - \dfrac{1}{LC}\right) V\omega}{L\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]} \sin(\omega t + \theta)$$

Putting
$$M \cos \phi = \frac{V \dfrac{\omega^2 R}{L^2}}{\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2}$$

and
$$M \sin \phi = \frac{V\left(\omega^2 - \dfrac{1}{LC}\right)\omega}{L\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]}$$

To find M and ϕ we divide one equation by the other

or
$$\frac{M \sin \phi}{M \cos \phi} = \tan \phi = \frac{\left(\omega L - \dfrac{1}{\omega C}\right)}{R}$$

$$\phi = \tan^{-1}\left[\left(\omega L - \frac{1}{\omega C}\right)\Big/R\right]$$

Squaring both equations and adding, we get

$$M^2 \cos^2 \phi + M^2 \sin^2 \phi = \frac{V^2}{R^2 + \left(\dfrac{1}{\omega C} - \omega L\right)^2}$$

$$M = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}}$$

The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\frac{\left(\frac{1}{\omega C} - \omega L\right)}{R}\right] \quad (11.41)$$

The complementary function is similar to that of DC series RLC circuit. To find out the complementary function, we have the characteristic equation

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) = 0 \quad (11.42)$$

The roots of Eq. 11.42, are

$$D_1, D_2 = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

By assuming $K_1 = -\frac{R}{2L}$ and $K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

$\therefore$ $D_1 = K_1 + K_2$ and $D_2 = K_1 - K_2$

K_2 becomes positive, when $(R/2L)^2 > 1/LC$

The roots are real and unequal, which gives an overdamped response. Then Eq. 11.42 becomes

$$[D - (K_1 + K_2)]\,[D - (K_1 - K_2)]i = 0$$

The complementary function for the above equation is

$$i_c = c_1 e^{(K_1 + K_2)t} + c_2 e^{(K_1 - K_2)t}$$

Therefore, the complete solution is

$$i = i_c + i_p$$

$$= c_1 e^{(K_1 + K_2)t} + c_2 e^{(K_1 - K_2)t}$$

$$+ \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

K_2 becomes negative, when $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$

Then the roots are complex conjugate, which gives an underdamped response. Equation 11.42 becomes

$$[D-(K_1+jK_2)][D-(K_1-jK_2)]i=0$$

The solution for the above equation is

$$i_c = e^{K_1 t}[c_1 \cos K_2 t + c_2 \sin K_2 t]$$

Therefore, the complete solution is

$$i = i_c + i_p$$

$$i = e^{K_1 t}[c_1 \cos K_2 t + c_2 \sin K_2 t]$$

$$+ \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

K_2 becomes zero, when $\left(\frac{R}{2L}\right)^2 = 1/LC$

Then the roots are equal which gives critically damped response. Then, Eq. 11.42 becomes $(D-K_1)(D-K_1)i=0$.

The complementary function for the above equation is

$$i_c = e^{K_1 t}(c_1 + c_2 t)$$

Therefore, the complete solution is $i = i_c + i_p$

$$\therefore \quad i = e^{K_1 t}[c_1 + c_2 t]$$

$$+ \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

Example 11.6 In the circuit shown in Fig. 11.21, determine the complete solution for the current, when the switch is closed at $t = 0$. Applied voltage is $v(t) = 400 \cos\left(500t + \frac{\pi}{4}\right)$. Resistance $R = 15\ \Omega$, inductance $L = 0.2$ H and capacitance $C = 3\mu$F.

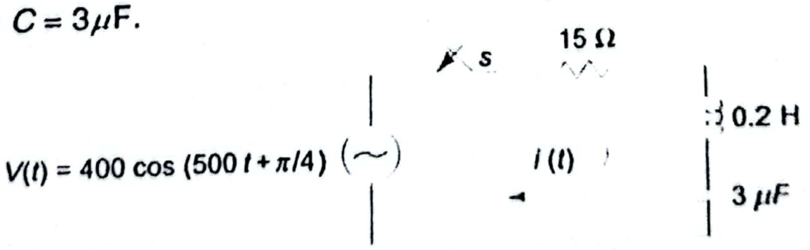

Fig. 11.21

Solution By applying Kirchhoff's voltage law to the circuit,

$$15i(t) + 0.2\frac{di(t)}{dt} + \frac{1}{3\times 10^{-6}}\int i(t)dt = 400 \cos\left(500t + \frac{\pi}{4}\right)$$

Differentiating the above equation once, we get

$$15\frac{di}{dt} + 0.2\frac{d^2i}{dt} + \frac{i}{3\times 10^{-6}} = -2\times 10^5 \sin\left(500t + \frac{\pi}{4}\right)$$

$$(D^2 + 75D + 16.7\times 10^5)i = \frac{-2\times 10^5}{0.2}\sin\left(500t + \frac{\pi}{4}\right)$$

The roots of the characteristic equation are

$$D_1 = -37.5 + j1290 \text{ and } D_2 = -37.5 - j1290$$

The complementary current

$$i_c = e^{-37.5t}(c_1 \cos 1290t + c_2 \sin 1290t)$$

Particular solution is

$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

$\therefore$

$$i_p = 0.71 \cos\left(500t + \frac{\pi}{4} + 88.5^\circ\right)$$

The complete solution is

$$i = e^{-37.5t}(c_1 \cos 1290t + c_2 \sin 1290t) + 0.71 \cos(500t + 45^\circ + 88.5^\circ)$$

At $t = 0$, $i_0 = 0$

$\therefore$

$$c_1 = -0.71 \cos(133.5^\circ) = +0.49$$

Differentiating the current equation, we have

$$\frac{di}{dt} = e^{-37.5t}(-1290c_1 \sin 1290t + c_2\, 1290 \cos 1290t)$$
$$- 37.5e^{-37.5t}(c_1 \cos 1290t + c_2 \sin 1290t)$$
$$- 0.71 \times 500 \sin(500t + 45^\circ + 88.5^\circ)$$

At $t = 0$, $\frac{di}{dt} = 1414$

$\therefore$

$$1414 = 1290c_2 - 37.5 \times 0.49 - 0.71 \times 500 \sin(133.5^\circ)$$
$$1414 = 1290c_2 - 18.38 - 257.5$$

$\therefore$

$$c_2 = 1.31$$

The complete solution is

$$i = e^{-37.5t}(0.49 \cos 1290t + 1.31 \sin 1290t) + 0.71 \cos(500t + 133.5^\circ)$$

Solved Problems

Problem 11.1 For the circuit shown in Fig. 11.22, find the current equation when the switch is changed from position 1 to position 2 at $t = 0$.

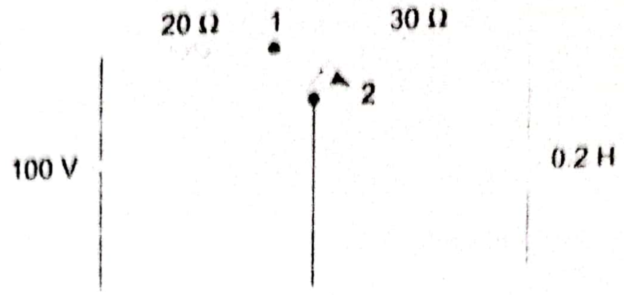

Fig. 11.22

Solution When the switch is at position 2, the current equation can be written by using Kirchhoff's voltage law as

$$30i(t) + 0.2\frac{di(t)}{dt} = 0$$

$$\left(D + \frac{30}{0.2}\right)i = 0$$

$$(D + 150)i = 0$$

$$i = c_1 e^{-150t}$$

At $t = 0$, the switch is changed to position 2, i.e. $i(0) = c_1$.

At $t = 0$, the initial current passing through the circuit is the same as the current passing through the circuit when the switch is at position 1. At $t = 0^-$, the switch is at position 1, and the current passing through the circuit $i = 100/50 = 2$ A.

At $t = 0^+$, the switch is at position 2. Since the inductor does not allow sudden changes in current, the same current passes through the circuit. Hence the initial current passing through the circuit, when the switch is at position 2 is $i(0^+) = 2$A.

$\therefore \quad c_1 = 2$ A

Therefore the current $i = 2e^{-150t}$

Problem 11.2 For the circuit shown in Fig. 11.23, find the current equation when the switch is opened at $t = 0$.

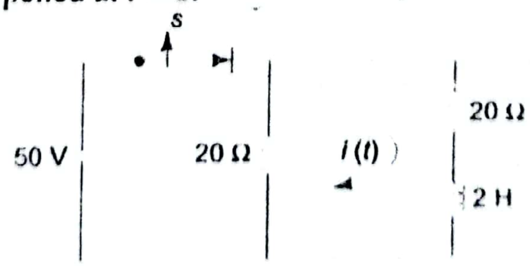

Fig. 11.23

Solution At $t = 0$, switch S is opened. By using Kirchhoff's voltage law, the current equation can be written as

$$20i + 20i + 2\frac{di}{dt} = 0$$

$$40i + 2\frac{di}{dt} = 0$$

$$D + 20i = 0$$

The solution for the above equation is

$$i = c_1 e^{-20t}$$

When the switch has been closed for a time, since the inductor acts as short circuit for dc voltages, the current passing through the inductor is 2.5 A.

That means, just before the switch is opened, the current passing through the inductor is 2.5 A. Since the current in the inductor cannot change instantaneously, $i(0^+)$ is also equal to 2.5 A.

At $t = 0$ $\quad c_1 = i(0^+) = 2.5$

Therefore, the final solution is $i(t) = 2.5e^{-20t}$

Problem 11.3 For the circuit shown in Fig. 11.24, find the current equation when the switch is opened at $t = 0$.

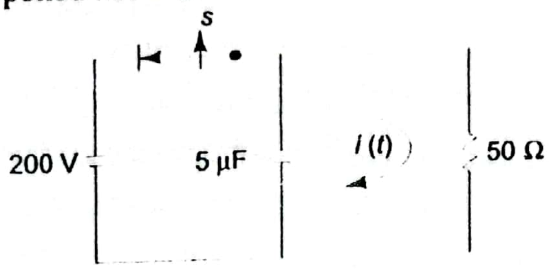

Fig. 11.24

Solution By using Kirchhoff's voltage law, the current equation is given by

$$\frac{1}{5 \times 10^{-6}} \int i\,dt + 50i = 0$$

Differentiating the above equation once, we get

$$50 \frac{di}{dt} + \frac{1}{5 \times 10^{-6}} i = 0$$

$$\therefore \quad \left(D + \frac{1}{250 \times 10^{-6}}\right) i = 0$$

$$\therefore \quad i = c_1 \exp\left(\frac{-1}{250 \times 10^{-6}} t\right) \qquad (11.43)$$

At $t = 0^-$, just before the switch S is opened, the voltage across the capacitor is 200 V. Since the voltage across the capacitor cannot change instantly, it remains equal to 200 V at $t = 0^+$. At that instant, the current through the resistor is

$$i(0^+) = \frac{200}{50} = 4\text{A}$$

In Eq. 11.43, the current is $i(0^+)$ at $t = 0$

$$\therefore \quad c_1 = 4 \text{ A}$$

Therefore, the current equation is

$$i = 4 \exp\left(\frac{-1}{250 \times 10^{-6}} t\right) \text{A}$$

Problem 11.4 For the circuit shown in Fig. 11.25, find the current equation when the switch S is opened at $t = 0$.

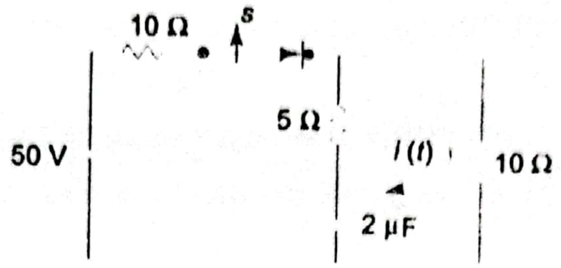

Fig. 11.25

Solution By using Kirchhoff's voltage law, the current equation is given by

$$\frac{1}{2 \times 10^{-6}} \int i dt + 5i + 10i = 0$$

Differentiating the above equation, we have

$$15 \frac{di}{dt} + \frac{i}{2 \times 10^{-6}} = 0$$

$$\left(D + \frac{1}{30 \times 10^{-6}}\right) i = 0$$

$$i = c_1 \exp\left(\frac{-1}{30 \times 10^{-6}}\right) t$$

At $t = 0^-$, just before switch S is opened, the current through 10 ohms resistor is 2.5 A. The same current passes through 10 Ω at $t = 0^+$

$$i(0^+) = 2.5 \text{ A}$$

At $t = 0$

$$i(0^+) = 2.5 \text{ A}$$

$$c_1 = 2.5$$

The complete solution is $i = 2.5 \exp\left(\frac{-1}{30 \times 10^{-6}} t\right)$

Problem 11.5 For the circuit shown in Fig. 11.26, find the complete expression for the current when the switch is closed at $t = 0$.

Solution By using Kirchhoff's law, the differential equation when the switch is closed at $t = 0$ is given by

$$20i + 0.1 \frac{di}{dt} = 100$$

$$(D + 200)i = 1000$$

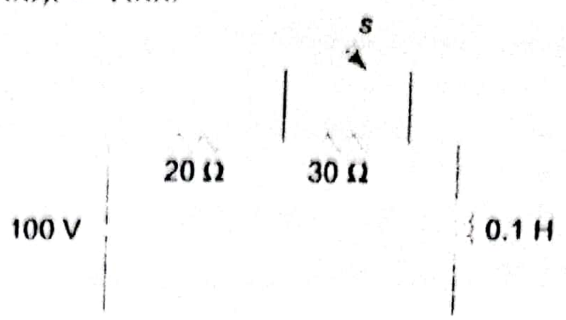

Fig. 11.26

$$i = c_1 e^{-200t} + e^{-200t} \int 1000 e^{200t}\, dt$$

$$\therefore \quad i = c_1\, e^{-200t} + 5$$

At $t = 0^-$, the current passing through the circuit is $i(0^-) = \dfrac{100}{50} = 2$ A. Since, the inductor does not allow sudden changes in currents, at $t = 0^+$, the same current passes through circuit.

$$\therefore \quad i(0^+) = 2 \text{ A}$$

At $t = 0$ $\quad i(0^+) = 2$

$$\therefore \quad c_1 = -3$$

The complete solution is $i = -3e^{-200t} + 5$ A

Problem 11.6 The circuit shown in Fig. 11.27, consists of series RL elements with $R = 150\ \Omega$ and $L = 0.5$ H. The switch is closed when $\phi = 30°$. Determine the resultant current when voltage $V = 50 \cos (100t + \phi)$ is applied to the circuit at $\phi = 30°$.

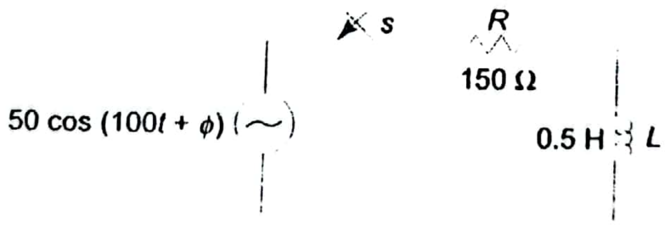

Fig. 11.27

Solution By using Kirchhoff's laws, the differential equation, when the switch is closed at $\phi = 30°$ is

$$150i + 0.5 \frac{di}{dt} = 50 \cos (100t + \phi)$$

$$0.5Di + 150i = 50 \cos (100t + 30°)$$

$$(D + 300)i = 100 \cos (100t + 30°)$$

The complementary current $i_c = ce^{-300t}$

To determine the particular current, first we assume a particular current

$$i_p = A \cos (100t + 30°) + B \sin (100t + 30°)$$

$$i'_p = -100\,A \sin(100t + 30^\circ) + 100\,B \cos(100t + 30^\circ)$$

Then

Substituting i_p and i'_p in the differential equation and equating the coefficients, we get

$$-100\,A \sin(100t + 30^\circ) + 100B \cos(100t + 30^\circ) + 300\,A \cos(100t + 30^\circ) + 300B \sin(100t + 30^\circ) = 100 \cos(100t + 30^\circ)$$

$$-100\,A + 300\,B = 0$$

$$300\,A + 100\,B = 100$$

From the above equation, we get

$$A = 0.3 \text{ and } B = 0.1$$

The particular current is

$$i_p = 0.3 \cos(100t + 30^\circ) + 0.1 \sin(100t + 30^\circ)$$

$$i_p = 0.316 \cos(100t + 11.57^\circ) \text{ A}$$

The complete equation for the current is $i = i_p + i_c$

$$i = ce^{-300t} + 0.316 \cos(100t + 11.57^\circ)$$

At $t = 0$, the current $i_0 = 0$

$$c = -0.316 \cos(11.57^\circ) = -0.309$$

Therefore, the complete solution for the current is

$$i = -0.309e^{-300t} + 0.316 \cos(100t + 11.57^\circ) \text{ A}$$

Problem 11.7 The circuit shown in Fig. 11.28, consists of series RC elements with $R = 15\ \Omega$ and $C = 100\ \mu$F. A sinusoidal voltage $v = 100 \sin(500t + \phi)$ volts is applied to the circuit at time corresponding to $\phi = 45^\circ$. Obtain the current transient.

Solution By using Kirchhoff's laws, the differential equation is

$$15i + \frac{1}{100 \times 10^{-6}} \int i\,dt = 100 \sin(500t + \phi)$$

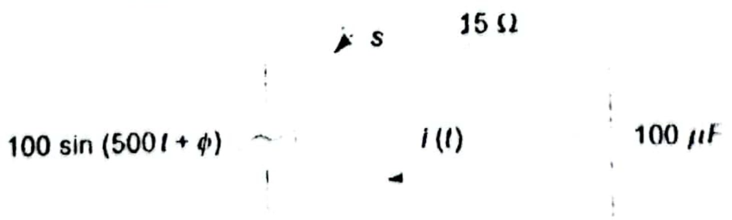

Fig. 11.28

Differentiating once, we have

$$15\frac{di}{dt} + \frac{1}{100 \times 10^{-6}}\, i = (100)\,(500) \cos(500t + \phi)$$

$$\left(D + \frac{1}{1500 \times 10^{-6}}\right) i = 3333.3 \cos(500t + \phi)$$

$$(D + 666.67)i = 3333.3 \cos(500t + \phi)$$

The complementary function $i_c = ce^{-666.67t}$

To determine the particular current, first we assume a particular current

$$i_p = A\cos(500t + 45°) + B\sin(500t + 45°)$$
$$i'_p = -500A\sin(500t + 45°) + 500B\cos(500t + 45°)$$

Substituting i_p and i'_p in the differential equation, we get

$$-500A\sin(500t + 45°) + 500B\cos(500t + 45°)$$
$$+ 666.67A\cos(500t + 45°) + 666.67B\sin(500t + 45°)$$
$$= 3333.3\cos(500t + \phi)$$

By equating coefficients, we get

$$500B + 666.67A = 3333.3$$
$$666.67B - 500A = 0$$

From which, the coefficients

$$A = 3.2;\ B = 2.4$$

Therefore, the particular current is

$$i_p = 3.2\cos(500t + 45°) + 2.4\sin(500t + 45°)$$
$$i_p = 4\sin(500t + 98.13°)$$

The complete equation for the current is

$$i = i_c + i_p$$
$$i = ce^{-666.67t} + 4\sin(500t + 98.13°)$$

At $t = 0$, the differential equation becomes

$$15i = 100\sin 45°$$
$$i = \frac{100}{15}\sin 45° = 4.71\ \text{A}$$

$\therefore$ At $t = 0$

$$4.71 = c + 4\sin(98.13°)$$

$\therefore$

$$c = 0.75$$

The complete current is

$$i = 0.75\,e^{-666.67t} + 4\sin(500t + 98.13°)$$

Problem 11.8 The circuit shown in Fig. 11.29 consisting of series RLC elements with $R = 10\ \Omega$, $L = 0.5$ H and $C = 200\ \mu$F has a sinusoidal voltage $v = 150\sin(200t + \phi)$. If the switch is closed when $\phi = 30°$, determine the current equation.

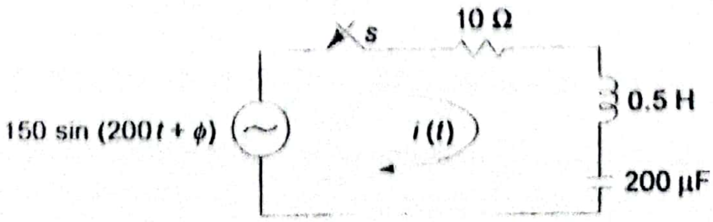

Fig. 11.29

Solution By using Kirchhoff's laws, the differential equation is

$$10i + 0.5\frac{di}{dt} + \frac{1}{200 \times 10^{-6}}\int i dt = 150 \sin(200t + \phi)$$

Differentiating once, we have

$$(D^2 + 20D + 10^4)i = 60000 \cos(200t + \phi)$$

The roots of the characteristics equation are

$$D_1 = -10 + j99.49 \text{ and } D_2 = -10 - j99.49$$

The complementary function is

$$i_c = e^{-10t}(c_1 \cos 99.49t + c_2 \sin 99.49)$$

We can find the particular current by using the undetermined coefficient method.

Let us assume

$$i_p = A \cos(200t + 30°) + B \sin(200t + 30°)$$
$$i'_p = -200 A \sin(200t + 30°) + 200 B \cos(200t + 30°)$$
$$i''_p = -(200)^2 A \cos(200t + 30°) - (200)^2 B \sin(220t + 30°)$$

Substituting these values in the equation, and equating the coefficients, we get

$$A = 0.1 \quad B = 0.067$$

Therefore, the particular current is

$$i_p = 1.98 \cos(200t - 52.4°) \text{ A}$$

The complete current is

$$i = e^{-10t}(c_1 \cos 99.49t + c_2 \sin 99.49t) + 1.98 \cos(200t - 52.4°) \text{ A}$$

From the differential equation at $t = 0$, $i_0 = 0$ and $\frac{di}{dt} = 300$

At $t = 0$

$$c_1 = -1.98 \cos(-52.4°) = -1.21$$

Differentiating the current equation, we have

$$\frac{di}{dt} = e^{-10t}(-99.49c_1 \sin 99.49t + 99.49c_2 \cos 99.49t)$$
$$-200(1.98)\sin(200t - 52.4°) - 10e^{-10t}(c_1 \cos 99.49t + c_2 \sin 99.49t)$$

At $t = 0$, $\frac{di}{dt} = 300$ and $c_1 = -1.21$

$$300 = 99.49\, c_2 - 396 \sin(-52.4°) - 10(-1.21)$$
$$300 = 99.49\, c_2 + 313.7 + 11.1$$
$$c_2 = -25.8$$

Therefore, the complete current equation is

$$i = e^{-10t}(0.07 \cos 99.49t - 25.8 \sin 99.49t) + 1.98 \cos(200t - 52.4°) \text{ A}$$

Problem 11.9 For the circuit shown in Fig. 11.30, determine the transient current when the switch is moved from position 1 to position 2 at $t = 0$. The circuit is in steady state with the switch in position 1. The voltage applied to the circuit is $v = 150 \cos(200t + 30°)$ V.

Fig. 11.30

Solution When the switch is at position 2, by applying Kirchhoff's law, the differential equation is

$$200i + 0.5\frac{di}{dt} = 0$$

$$(D + 400)i = 0$$

∴ The transient current is

$$i = ce^{-400t}$$

At $t = 0$, the switch is moved from position 1 to position 2. Hence the current passing through the circuit is the same as the steady state current passing through the circuit when the switch is in position 1.

When the switch is in position 1, the current passing through the circuit is

$$i = \frac{v}{z} = \frac{150\angle 30°}{R + j\omega L}$$

$$= \frac{150\angle 30°}{200 + j(200)(0.5)} = \frac{150\angle 30°}{223.6\angle 26.56°} = 0.67\angle 3.44°$$

Therefore, the steady state current passing through the circuit when the switch is in position 1 is

$$i = 0.67\cos(200t + 3.44°)$$

Now substituting this equation in transient current equation, we get

$$0.67\cos(200t + 3.44°) = ce^{-400t}$$

At $t = 0$; $c = 0.67\cos(3.44°) = 0.66$

Therefore, the current equation is $i = 0.66e^{-400t}$

Problem 11.10 In the circuit shown in Fig. 11.31, determine the current equations for i_1 and i_2 when the switch is closed at $t = 0$.

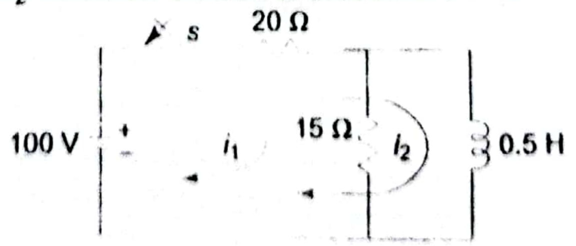

Fig. 11.31

Solution By applying Kirchhoff's laws, we get two equations

$$35i_1 + 20i_2 = 100 \quad (11.44)$$

$$20i_1 + 20i_2 + 0.5\frac{di_2}{dt} = 100 \quad (11.45)$$

From Eq. 11.44, we have

$$35i_1 = 100 - 20i_2$$

$$i_1 = \frac{100}{35} - \frac{20}{35}i_2$$

Substituting i_1 in Eq. 11.45, we get

$$20\left(\frac{100}{35} - \frac{20}{35}i_2\right) + 20i_2 + 0.5\frac{di_2}{dt} = 100 \quad (11.46)$$

$$57.14 - 11.43i_2 + 20i_2 + 0.5\frac{di_2}{dt} = 100$$

$$(D + 17.14)i_2 = 85.72$$

From the above equation,

$$i_2 = ce^{-17.14t} + 5$$

Loop current i_2 passes through inductor and must be zero at $t = 0$

At $t = 0$, $i_2 = 0$

∴ $c = -5$

∴ $i_2 = 5(1 - e^{-17.14t})$ A

and the current $i_1 = 2.86 - \{0.57 \times 5(1 - e^{-17.14t})\}$

$= (0.01 + 2.85\, e^{-17.14t})$ A

Problem 11.11 For the circuit shown in Fig. 11.32, find the current equation when the switch is changed from position 1 to position 2 at $t = 0$.

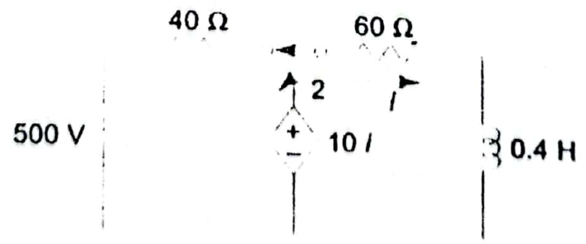

Fig. 11.32

Solution By using Kirchhoff's voltage law, the current equation is given by

$$60i + 0.4\frac{di}{dt} = 10i$$

At $t = 0^-$, the switch is at position 1, the current passing through the circuit is

$$i(0^-) = \frac{500}{100} = 5 \text{ A}$$

$$0.4\frac{di}{dt} + 50i = 0$$

$$\left(D + \frac{50}{0.4}\right)i = 0$$

$$i = ce^{-125t}$$

At $t = 0$, the initial current passing through the circuit is same as the current passing through the circuit when the switch is at position 1.

At $t = 0$, $i(0) = i(0^-) = 5$ A

At $t = 0$, $c = 5$ A

∴ The current $I = 5e^{-125t}$

Problem 11.12 For the circuit shown in Fig. 11.33, find the current equation when the switch S is opened at $t = 0$.

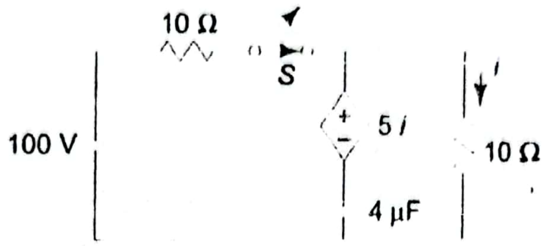

Fig. 11.33

Solution When the switch is closed for a long time,

At $t = 0^-$, the current $i(0^-) = \frac{100}{20} = 5$ A

When the switch is opened at $t = 0$, the current equation by using Kirchhoff's voltage law is given by

$$\frac{1}{4\times10^{-6}}\int i\,dt + 10i = 5i$$

$$\frac{1}{4\times10^{-6}}\int i\,dt + 5i = 0$$

Differentiating the above equation

$$5\frac{di}{dt} + \frac{1}{4\times10^{-6}}i = 0$$

$$\left(D + \frac{1}{20\times10^{-6}}\right)i = 0$$

$$\therefore \quad i = ce^{\frac{-1}{20\times10^{-6}}t}$$

At $t = 0^-$, just before switch S is opened, the current passing through the 10 Ω resistor is 5 A. The same current passes through 10 Ω at $t = 0$.

$\therefore$ At $t = 0, i(0) = 5\ A$

At $t = 0, c_1 = 5\ A$

The current equation is $i = 5e^{\frac{-t}{20 \times 10^{-6}}}$

Problem 11.13 For the circuit shown in Fig. 11.34, find the current in the 20 Ω when the switch is opened at $t = 0$.

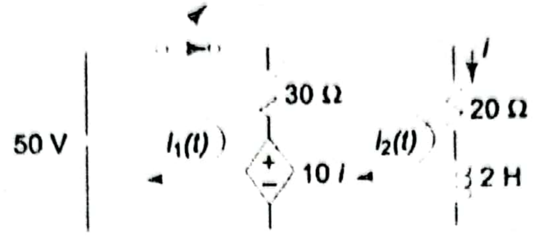

Fig. 11.34

Solution When the switch is closed, the loop current i_1 and i_2 are flowing in the circuit.

The loop equations are $30(i_1 - i_2) + 10i_2 = 50$

$$30(i_2 - i_1) + 20i_2 = 10i_2$$

From the above equations, the current in the 20 Ω resistor $i_2 = 2.5$ A.

The same initial current is flowing when the switch is opened at $t = 0$.

When the switch is opened the current equations

$$30i + 20i + 2\frac{di}{dt} = 10i$$

$$40i + \frac{2di}{dt} = 0$$

$$(D + 20)i = 0$$

$$i = ce^{-20t}$$

At $t = 0$, the current $i(0) = 2.5$ A

$\therefore$ At $t = 0$, $c = 2.5$

The current in the 20 Ω resistor is $i = 2.5\ e^{-20t}$.

Problem 11.14 For the circuit shown in Fig. 11.35, find the current equation when the switch is opened at $t = 0$.

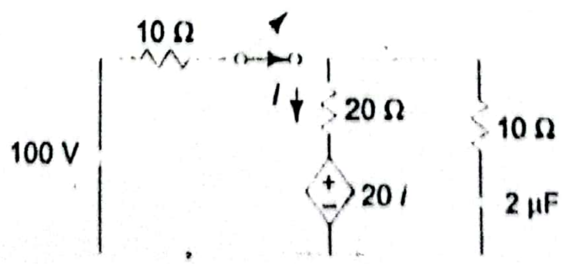

Fig. 11.35

Solution When the switch is closed, the current in the 20 Ω resistor i can be obtained using Kirchhoff's voltage law.

$$10i + 20i + 20i = 100$$

$$50i = 100, \therefore i = 2 \text{ A}$$

The same initial current passes through the 20 Ω resistor when the switch is opened at $t = 0$.

The current equation is

$$20i + 10i + \frac{1}{2 \times 10^{-6}} \int i dt = 20i$$

$$10i + \frac{1}{2 \times 10^{-6}} \int i dt = 0$$

Differentiating the above equation, we get

$$10\frac{di}{dt} + \frac{1}{2 \times 10^{-6}} i = 0$$

$$\left(D + \frac{1}{20 \times 10^{-6}}\right) i = 0$$

The solution for the above equation is

$$i = ce^{\frac{-1}{20 \times 10^{-6}} t}$$

At $t = 0$, $i(0) = i(0^-) = 2$ A

∴ At $t = 0$, $c = 2$ A

The current equation is

$$i = 2e^{\frac{-1}{20 \times 10^{-6}} t}$$

Practice Problems

11.1 (a) What do you understand by transient and steady state parts of response? How can they be identified in a general solution?

(b) Obtain an expression for the current $i(t)$ from the differential equation

$$\frac{d^2 i(t)}{dt^2} + 10 \frac{di(t)}{dt} + 25 i(t) = 0$$

with initial conditions

$$i(0^+) = 2 \quad \frac{di(0^+)}{dt} = 0$$

11.2 A series circuit shown in Fig. 11.36, comprising resistance 10 Ω and inductance 0.5 H, is connected to a 100 V source at $t = 0$. Determine the complete expression for the current $i(t)$.

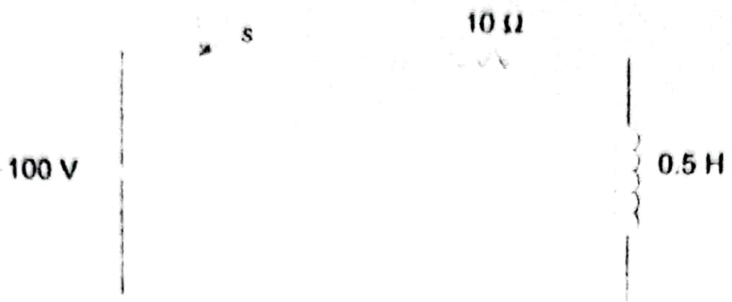

Fig. 11.36

11.3 In the network shown in Fig. 11.37, the capacitor c_1 is charged to a voltage of 100 V and the switch S is closed at $t = 0$. Determine the current expression i_1 and i_2.

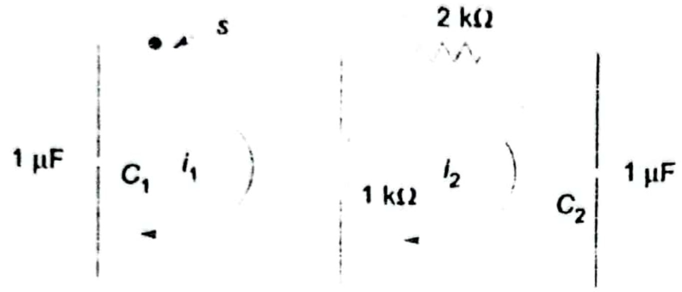

Fig. 11.37

11.4 A series RLC circuit shown in Fig. 11.38, comprising $R = 10\,\Omega$, $L = 0.5$ H and $C = 1\,\mu$F, is excited by a constant voltage source of 100 V. Obtain the expression for the current. Assume that the circuit is relaxed initially.

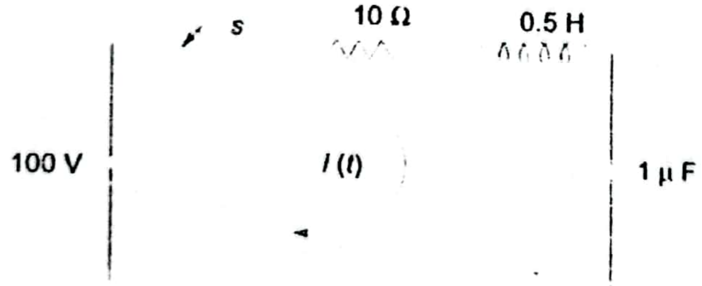

Fig. 11.38

11.5 In the circuit shown in Fig. 11.39, the initial current in the inductance is 2 A and its direction is as shown in the figure. The initial charge on the capacitor is 200 C with polarity as shown when the switch is closed. Determine the current expression in the inductance.

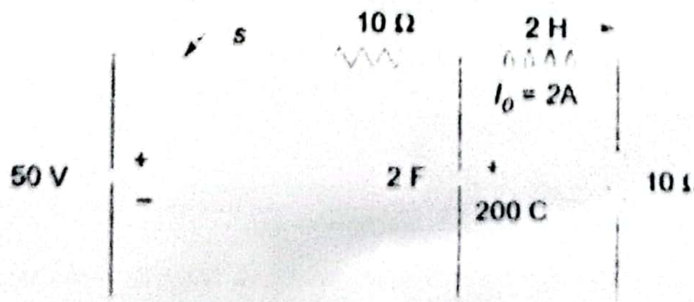

Fig. 11.39